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Large deviations of the empirical characteristic function

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1. Introduction. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) random variables defined on a probability space $(\Omega, \mathfrak{A}, P)$ and taking values in $\mathbf{R}$ with common distribution function $F(x)$, $x \in \mathbf{R}$, and characteristic function

$$c(t) = \int_{\mathbf{R}} e^{itx}\, dF(x), \quad t \in \mathbf{R}.$$

The n^{th} empirical characteristic function (e.c.f.) of the sequence is

$$c_n(t) = (1/n) \sum_{j=1}^{n} e^{itX_j} = \int_{\mathbf{R}} e^{itx}\, dF_n(x), \quad t \in \mathbf{R},$$

where $F_n(x)$, $x \in \mathbf{R}$, denotes the empirical distribution function (e.d.f.) of $X_1, \ldots, X_n$. Csörgő [2], [3] and Marcus [8] gave necessary and sufficient conditions for the weak convergence of the empirical characteristic process $\sqrt{n}(c_n(t)-c(t))$ in the space of continuous complex-valued functions on a compact interval. Csörgő and Totik [4] solved the problem of consistency. The present investigation deals with the problem of large deviations of the e.c.f.. More precisely, let $S \subset \mathbf{R}$ and $T_n = \sup_{t \in S} |c_n(t)-c(t)|$, $n \in \mathrm{N}$. We shall derive asymptotic expressions for the limit

$$\lim_{n\to\infty} (1/n) \log P\{T_n \geqq \varepsilon\}, \quad \varepsilon > 0.$$

Theorems on probabilities of large deviations for related statistics are contained in the work of Groeneboom (see e.g. [6]) and many other authors, a powerful theory being available now. But such results only yield first order terms in an expansion of logarithms of large deviation probabilities, whereas our representation immediately gives higher order terms and can be used for the computation of the (relative) asymptotic Bahadur efficiency. Although some doubt exists as to the value of the concept of Bahadur efficiency, the present work was partly motivated by it (cf. [7]).

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2. Results. If $p\in[0,+\infty)$, $\varepsilon\in(0,1)$, let $J(0,\varepsilon)=+\infty$, $J(1-\varepsilon,\varepsilon)=-\log(1-\varepsilon)$,

$$J(p,\varepsilon)=$$
$$=\begin{cases}(p+\varepsilon)\log((p+\varepsilon)/p)+(1-p-\varepsilon)\log((1-p-\varepsilon)/(1-p)) & \text{if } 0<p<1-\varepsilon\\ +\infty & \text{if } 1-\varepsilon<p.\end{cases}$$

Lemma 1. *Suppose* $\varepsilon\in(0,1)$. *Then*

$$\overline{\lim_{n\to\infty}}\,(1/n)\log P\{\sup_{x\in\mathbf{R}}|F_n(x)-F(x)|\geqq\varepsilon\}\leqq-\min\{J(p,\varepsilon)\colon 0<p\leqq 1-\varepsilon\}.$$

Proof. Let $U_1, U_2, \ldots$ be a sequence of i.i.d. $U(0,1)$ random variables defined on a probability space $(\Omega^*, \mathfrak{A}^*, P^*)$. Denote the e.d.f. of the sample $U_1, \ldots, U_n$ by G_n. If $u\in[0,1]$ let $F^{-1}(u)=\inf\{x\colon F(x)\geqq u\}$. Then $F^{-1}(u)\leqq x$ if and only if $u\leqq F(x)$. X_1 and $F^{-1}(U_1)$ are identically distributed. Hence we get

$$P\{\sup_{x\in\mathbf{R}}|F_n(x)-F(x)|\geqq\varepsilon\}=P^*\{\sup_{x\in\mathbf{R}}|G_n(F(x))-F(x)|\geqq\varepsilon\}\leqq$$
$$\leqq P^*\{\sup_{0\leqq x\leqq 1}|G_n(x)-x|\geqq\varepsilon\}.$$

This completes our proof since

$$\lim_{n\to\infty}(1/n)\log P^*\{\sup_{0\leqq x\leqq 1}|G_n(x)-x|\geqq\varepsilon\}=-\min\{J(p,\varepsilon)\colon 0<p\leqq 1-\varepsilon\}$$

(cf. [6], Example 1.3.1, p. 21).

Before stating our main theorem let us introduce the random vector $Y_j(t)=(\cos(tX_j)-\operatorname{Re}c(t),\ \sin(tX_j)-\operatorname{Im}c(t))$ with its Laplace transform $M_t(\theta)=\int\exp(\langle\theta, Y_1(t)\rangle)dP$, $\theta\in\mathbf{R}^2$, for $t\in\mathbf{R}$, $j\in\mathbf{N}$. If $t\in\mathbf{R}$, $\varepsilon>0$, $\theta\in\mathbf{R}^2$, let $h_t(\varepsilon,\theta)=\inf\{\exp(-r\varepsilon)M_t(r\theta)\colon r\geqq 0\}$ and if $t\in\mathbf{R}$, $\varepsilon>0$, let $i_t(\varepsilon)=\log(\sup\{h_t(\varepsilon,\theta)\colon \theta\in\mathbf{R}^2, \|\theta\|=1\})$. Let $C(S)$ be the space of continuous functions on S, AP the space of all almost periodic functions.

Now the following theorem holds:

Theorem 2. *Let the subset* S *be compact and let* $i(\varepsilon)=\sup\{i_t(\varepsilon)\colon t\in S\}$ *for each* $\varepsilon>0$. *Then* $\lim_{n\to\infty}(1/n)\log P\{T_n\geqq\varepsilon\}=i(\varepsilon)$.

Proof. *If* $t\in S$, we get by Theorem 7 of Sethuraman [10] that

$$i_t(\varepsilon)=\lim_{n\to\infty}(1/n)\log P\{|c_n(t)-c(t)|\geqq\varepsilon\}\leqq\underline{\lim}_{n\to\infty}(1/n)\log P\{T_n\geqq\varepsilon\}.$$

Hence $i(\varepsilon)\leqq\underline{\lim}_{n\to\infty}(1/n)\log P\{T_n\geqq\varepsilon\}$.

Now let $k\in\mathbf{N}$ be arbitrary, and let us cover the set S by a finite number of open balls $B(k_j, 1/k)$ with center $k_j\in S$ and radius $1/k$, $1\leqq j\leqq k_*$. Writing

$$S_{n,k}=\sup\{|c_n(t)-c(t)-(c_n(t^*)-c(t^*))|\colon t, t^*\in S, |t-t^*|<1/k\},$$

we have

$$T_n \leqq \max_{1\leqq j\leqq k_*} |c_n(k_j)-c(k_j)|+S_{n,k} \quad \text{for each} \quad n\in\mathbf{N}.$$

Let $\delta\in(0, \varepsilon)$ be given. Then

$$\overline{\lim_{n\to\infty}}\, (1/n)\log P\{T_n \geqq \varepsilon\} \leqq$$

$$\leqq \overline{\lim_{n\to\infty}}\, (1/n)\log \big[P\big\{\max_{1\leqq j\leqq k_*} |c_n(k_j)-c(k_j)|+S_{n,k} \geqq \varepsilon, \delta > S_{n,k}\big\}+$$

$$+P\big\{\max_{1\leqq j\leqq k_*} |c_n(k_j)-c(k_j)|+S_{n,k} \geqq \varepsilon, S_{n,k} \geqq \delta\big\}\big] \leqq$$

$$\leqq \overline{\lim_{n\to\infty}}\, (1/n)\log \big[P\big\{\max_{1\leqq j\leqq k_*} |c_n(k_j)-c(k_j)| \geqq \varepsilon-\delta\big\}+P\{S_{n,k} \geqq \delta\}\big] \leqq$$

$$\leqq \overline{\lim_{n\to\infty}}\, (1/n)\log \big[2\cdot \max\big\{P\big\{\max_{1\leqq j\leqq k_*} |c_n(k_j)-c(k_j)| \geqq \varepsilon-\delta\big\}, P\{S_{n,k} \geqq \delta\}\big\}\big] =$$

$$= \max\big\{\overline{\lim_{n\to\infty}}\, (1/n)\log P\big\{\max_{1\leqq j\leqq k_*} |c_n(k_j)-c(k_j)| \geqq \varepsilon-\delta\big\}, \overline{\lim_{n\to\infty}}\, (1/n)\log P\{S_{n,k} \geqq \delta\}\big\}.$$

Looking for a bound for the first $\overline{\lim}$ in this expression, we get

$$\overline{\lim_{n\to\infty}}\, (1/n)\log P\big\{\max_{1\leqq j\leqq k_*} |c_n(k_j)-c(k_j)| \geqq \varepsilon-\delta\big\} \leqq$$

$$\leqq \overline{\lim_{n\to\infty}}\, (1/n)\log \sum_{j=1}^{k_*} P\{|c_n(k_j)-c(k_j)| \geqq \varepsilon-\delta\} \leqq \max_{1\leqq j\leqq k_*} i_{k_j}(\varepsilon-\delta) \leqq i(\varepsilon-\delta).$$

The second $\overline{\lim}$ requires some computations concerning $S_{n,k}$. Let $t, t^*\in S$, $|t-t^*|< 1/k$ and $\lambda>0$ be given, where λ is a continuity point of F such that $\delta/16> 1-F(\lambda)+F(-\lambda)$. Then we get

$$\big|c_n(t)-c(t)-\big(c_n(t^*)-c(t^*)\big)\big| \leqq$$

$$\leqq \Big|\int_{\{|x|>\lambda\}} (e^{itx}-e^{it^*x})\, d\big(F_n(x)-F(x)\big)\Big| + \Big|\int_{\{|x|\leqq\lambda\}} (e^{itx}-e^{it^*x})\, d\big(F_n(x)-F(x)\big)\Big|.$$

Now

$$\Big|\int_{\{|x|>\lambda\}} (e^{itx}-e^{it^*x})\, d\big(F_n(x)-F(x)\big)\Big| \leqq$$

$$\leqq \int_{\{|x|>\lambda\}} |e^{itx}-e^{it^*x}|\, dF_n(x) + \int_{\{|x|>\lambda\}} |e^{itx}-e^{it^*x}|\, dF(x) \leqq$$

$$\leqq 2|F_n(\lambda)-F(\lambda)|+2|F_n(-\lambda)-F(-\lambda)|+4\big(1-F(\lambda)+F(-\lambda)\big).$$

Let $K=\max\{|t|: t\in S\}$ and $\lambda^*=2(1+K\lambda)\lambda$. Using integration by parts,

$$\Big|\int_{-\lambda}^{+\lambda}(e^{itx}-e^{it^*x})\,d\big(F_n(x)-F(x)\big)\Big| =$$

$$= \Big|(e^{it\lambda}-e^{it^*\lambda})\big(F_n(\lambda-)-F(\lambda-)\big)-(e^{it(-\lambda)}-e^{it^*(-\lambda)})\big(F_n(-\lambda)-F(-\lambda)\big)-$$

$$-i\int_{-\lambda}^{+\lambda}(F_n(x)-F(x))(te^{itx}-t^*e^{it^*x})\,dx\Big| \leqq$$

$$\leqq 2|F_n(\lambda-)-F(\lambda)|+2|F_n(-\lambda)-F(-\lambda)|+|t-t^*|\lambda^*\cdot\sup_{x\in\mathbf{R}}|F_n(x)-F(x)| \leqq$$

$$\leqq 2|F_n(\lambda-)-F(\lambda)|+2|F_n(-\lambda)-F(-\lambda)|+(\lambda^*/k)\sup_{x\in\mathbf{R}}|F_n(x)-F(x)|.$$

Summing up

$$S_{n,k} \leqq 2|F_n(\lambda)-F(\lambda)|+2|F_n(\lambda-)-F(\lambda)|+4|F_n(-\lambda)-F(-\lambda)|+$$

$$+4\big(1-F(\lambda)+F(-\lambda)\big)+(\lambda^*/k)\sup_{x\in\mathbf{R}}|F_n(x)-F(x)|.$$

Hence $\overline{\lim}_{n\to\infty} 1/n \log P\{T_n\geqq\varepsilon\}$ is bounded by the maximum of $i(\varepsilon-\delta)$,

$$\overline{\lim_{n\to\infty}}\,(1/n)\log P\{|F_n(\lambda)-F(\lambda)|\geqq\delta/16\},\quad \overline{\lim_{n\to\infty}}\,(1/n)\log P\{|F_n(\lambda-)-F(\lambda)|\geqq\delta/8\},$$

$$\overline{\lim_{n\to\infty}}\,(1/n)\log P\{|F_n(-\lambda)-F(-\lambda)|\geqq\delta/16\}\quad\text{and}$$

$$\overline{\lim_{n\to\infty}}\,(1/n)\log P\Big\{\sup_{x\in\mathbf{R}}|F_n(x)-F(x)|\geqq(k/\lambda^*)\big(\delta/4-4(1-F(\lambda)+F(-\lambda))\big)\Big\}.$$

If we let first k and then λ tend to infinity, we get

$$\overline{\lim_{n\to\infty}}\,(1/n)\log P\{T_n\geqq\varepsilon\}\leqq i(\varepsilon-\delta).$$

This can be seen from the equality

$$\overline{\lim_{n\to\infty}}\,(1/n)\log P\{|F_n(x)-F(x)|\geqq\varepsilon\} =$$

$$= -\min\big\{J(F(x),\varepsilon),\,J\big(1-F(x),\varepsilon\big)\big\},\quad x\in\mathbf{R},\quad \varepsilon>0,$$

and Lemma 1 or directly.

Finally, $\delta\in(0,\varepsilon)$ was arbitrary. Hence i is continuous from the left by [10] Theorem 7 $(\mathscr{X}=C(S))$ and [10] Lemma 3. This gives the result.

Example 3. If $L(X_1)=B(1,p)$, i.e. $P\{X_1=0\}=p$ and $P(X_1=1\}=1-p=:q$, some straightforward computations lead to the equality

$$i_t(\varepsilon)=\begin{cases}\max\{-J(p,\varepsilon/a_t),\ -J(q,\varepsilon/a_t)\} & \text{if}\quad \varepsilon<a_t\\ -\infty & \text{otherwise,}\end{cases}$$

where $a_t=(2(1-\cos t))^{1/2}$. Hence $i(\varepsilon)=\max\{-J(p, \varepsilon/a), -J(q, \varepsilon/a)\}$, if $T>0$, $S=[-T, +T]$ and $a=\max\{a_t\colon t\in S\}>\varepsilon$.

That T_n converges to zero almost surely even in the case $S=\mathbf{R}$ when F is purely discrete was pointed out by FEUERVERGER and MUREIKA [5]. We are now able to derive the corresponding large deviation generalization of Theorem 2.

Theorem 4. *Let F be purely discrete. If $S=\mathbf{R}$ and $i(\varepsilon)=\sup\{i_t(\varepsilon)\colon t\in S\}>-\infty$ for each $\varepsilon>0$, then $\lim_{n\to\infty}(1/n)\log P\{T_n\geqq\varepsilon\}=i(\varepsilon)$.*

Proof. With the same conclusion as in the proof of Theorem 2 we get $i(\varepsilon)\leqq\varliminf_{n\to\infty}(1/n)\log P\{T_n\geqq\varepsilon\}$. Now F is purely discrete. Hence there exist $N\in\mathbf{N}\cup\{+\infty\}$, $p_k\geqq 0$ and pairwise distinct $a_k\in\mathbf{R}$ with $P\{X_1=a_k\}=p_k$, $k\in\mathbf{N}$, $1\leqq k<N+1$, and $\sum_{k=1}^{N}p_k=1$. Let $\delta\in(0,\varepsilon)$, $m\in\mathbf{N}$, $m<N+1$, and $\gamma>0$ be given and let f denote the function $f(t)=\sum_{k=1}^{m}|1-e^{ita_k}|^2$, $t\in\mathbf{R}$. Since f is almost periodic, there exists an $L=L(\gamma^2)>0$ such that every interval of the real axis of length not smaller than L contains at least one ε-almost period, i.e. a number τ satisfying $|f(t+\tau)-f(t)|<\gamma^2$ for all $t\in\mathbf{R}$. Hence, if $t\in\mathbf{R}$ is fixed now, we can choose an ε-almost period from the open interval $(-t, -t+L)$. Then we get

$$|c_n(t)-c(t)|\leqq|c_n(t+\tau)-c(t+\tau)|+\big|c_n(t)-c(t)-\big(c_n(t+\tau)-c(t+\tau)\big)\big|\leqq$$
$$\leqq\sup_{t\in(0,L)}|c_n(t)-c(t)|+\big|c_n(t)-c(t)-\big(c_n(t+\tau)-c(t+\tau)\big)\big|.$$

It follows from Theorem 2 that

$$\varlimsup_{n\to\infty}(1/n)\log P\Big\{\sup_{t\in(0,L)}|c_n(t)-c(t)|\geqq\varepsilon-\delta\Big\}\leqq i(\varepsilon-\delta).$$

Next we study the second term which is a.s.

$$\Big|(1/n)\sum_{j=1}^{n}\sum_{k=1}^{N}\big((e^{ita_k}-e^{i(t+\tau)a_k})(I_{\{X_j=a_k\}}-p_k)\big)\Big|\leqq$$

$$\leqq(1/n)\sum_{j=1}^{n}\sum_{k=1}^{N}|e^{ita_k}-e^{i(t+\tau)a_k}|\cdot|I_{\{X_j=a_k\}}-p_k|\leqq$$

$$\leqq(1/n)\sum_{j=1}^{n}\Big[\sum_{k=1}^{m}|e^{ita_k}-e^{i(t+\tau)a_k}|\cdot|I_{\{X_j=a_k\}}-p_k|+2\sum_{k=m+1}^{N}|I_{\{X_j=a_k\}}-p_k|\Big]\leqq$$

$$\leqq(1/n)\sum_{j=1}^{n}\Big[\Big(f(\tau)\cdot\sum_{k=1}^{m}|I_{\{X_j=a_k\}}-p_k|^2\Big)^{1/2}+2\sum_{k=m+1}^{N}|I_{\{X_j=a_k\}}-p_k|\Big]\leqq$$

$$\leqq(1/n)\sum_{j=1}^{n}\Big[\gamma\Big(\sum_{k=1}^{m}|I_{\{X_j=a_k\}}-p_k|^2\Big)^{1/2}+2\sum_{k=m+1}^{N}|I_{\{X_j=a_k\}}-p_k|\Big]=(1/n)\sum_{j=1}^{n}Z_j,$$

where

$$Z_j = \gamma\Big(\sum_{k=1}^{m} |I_{\{X_j=a_k\}}-p_k|^2\Big)^{1/2}+2\sum_{k=m+1}^{N} |I_{\{X_j=a_k\}}-p_k|,$$

$1\leqq j\leqq n$. It follows from Theorem 3.1 of BAHADUR [1] that

$$\overline{\lim_{n\to\infty}}\,(1/n)\log P\Big\{(1/n)\sum_{j=1}^{n} Z_j\geqq\delta\Big\} = \inf\Big\{\Big(\log\int e^{rZ_1}\,dP\Big)-r\delta\colon r\geqq 0\Big\}.$$

But now, since $t\in\mathbf{R}$ was arbitrary, the preceding inequalities lead to

$$\overline{\lim_{n\to\infty}}\,(1/n)\log P\Big\{\sup_{t\in\mathbf{R}}|c_n(t)-c(t)|\geqq\varepsilon\Big\}\leqq$$

$$\leqq \max\Big\{i(\varepsilon-\delta),\ \Big(\log\int e^{rZ_1}\,dP\Big)-r\delta\Big\}\quad\text{for all}\quad r\geqq 0.$$

If we let first γ converge to zero, then m tend to N, and finally r go to $+\infty$, we get

$$\overline{\lim_{n\to\infty}}\,(1/n)\log P\{T_n\geqq\varepsilon\}\leqq i(\varepsilon-\delta).$$

The closing step in the proof of Theorem 2 yields the desired result ($\mathscr{X}\subset AP$!).

Having Theorems 2 and 4, we are finally interested in an expansion of the limits of the logarithmic large deviation probabilities.

Lemma 5. *Let* $t\in\mathbf{R}$, $A_t=E\cos(tX_1)$, $B_t=E\sin(tX_1)$, $C_t=E\cos^2(tX_1)$, $D_t=E(\sin tX_1\cdot\cos tX_1)$, $E_t=E\sin^2(tX_1)$ *and*

$$\sigma_t^2 = (1/2)[(C_t-A_t^2)+(E_t-B_t^2)]+\big[(1/4)((C_t-A_t^2)-(E_t-B_t^2))^2+(D_t-A_tB_t)^2\big]^{1/2}.$$

Then $\sigma_t^2=\sup\{\mathrm{Var}\langle\theta, Y_1(t)\rangle\colon \theta\in\mathbf{R}^2,\ \|\theta\|=1\}$ *and*

$$i_t(\varepsilon) = -\varepsilon^2/2\sigma_t^2+o(\varepsilon^2)\quad as\quad \varepsilon\downarrow 0.$$

Proof. For $\theta=(\theta_1,\theta_2)\in\mathbf{R}^2$ we have

$$\mathrm{Var}\langle\theta, Y_1(t)\rangle = E\big((\theta_1(\cos(tX_1)-\mathrm{Re}\,c(t))+\theta_2(\sin(tX_1)-\mathrm{Im}\,c(t)))^2\big).$$

Defining $a=E(\cos(tX_1)-\mathrm{Re}\,c(t))^2$, $b=E\big((\cos(tX_1)-\mathrm{Re}\,c(t))(\sin(tX_1)-\mathrm{Im}\,c(t))\big)$, $c=E(\sin(tX_1)-\mathrm{Im}\,c(t))^2$ and

$$\mathscr{B}=\begin{pmatrix} a & b\\ b & c\end{pmatrix},$$

we get $\mathrm{Var}\langle\theta, Y_1(t)\rangle=\theta\mathscr{B}\theta^{\mathrm{T}}$. Hence $\sup\{\mathrm{Var}\langle\theta, Y_1(t)\rangle\colon\theta\in\mathbf{R}^2,\ \|\theta\|=1\}$ is the greatest eigenvalue of $\mathscr{B}$, which is equal to

$$(1/2)(a+c)+[(1/4)(a-c)^2+b^2]^{1/2}.$$

This proves the first part of our lemma, since $a=C_t-A_t^2$, $b=D_t-A_tB_t$ and $c=E_t-B_t^2$. The remaining expansion follows from Lemma 2.2 of JAMMALAMADAKA RAO [9].

Now Lemma 5 immediately yields expansions for the functions i appearing in our Theorems 2 and 4.

Theorem 6. *Let* $S\subset\mathbf{R}$ *be arbitrary,* $i(\varepsilon)=\sup\{i_t(\varepsilon)\colon t\in S\}$ *and* $\sigma^2:=\sup\{\sigma_t^2\colon t\in S\}$, *where* $0<\sigma^2<+\infty$. *Then* $i(\varepsilon)=-\varepsilon^2/2\sigma^2+o(\varepsilon^2)$ *as* $\varepsilon\downarrow 0$.

Proof. Let $(\varepsilon_n)_{n\in\mathbf{N}}$ be a sequence of positive real numbers converging to zero. Then Lemma 5 yields

$$-1/2\sigma_t^2 = \lim_{n\to\infty} i_t(\varepsilon_n)/\varepsilon_n^2 \leqq \varliminf_{n\to\infty} i(\varepsilon_n)/\varepsilon_n^2 \quad \text{for all} \quad t\in S.$$

This implies $-1/2\sigma^2\leqq\varliminf\limits_{n\to\infty} i(\varepsilon_n)/\varepsilon_n^2$. Now

$$i_t(\varepsilon) = \log\Big(\sup\Big\{\inf_{r\geqq 0}\exp(-r\varepsilon)\int e^{r\langle\theta, Y_1(t)\rangle}\,dP\colon \theta\in\mathbf{R}^2, \|\theta\| = 1\Big\}\Big) =$$

$$= \sup\Big\{\inf\Big\{-r\varepsilon+\log\int e^{r\langle\theta, Y_1(t)\rangle}\,dP\colon r\geqq 0\Big\}\colon \theta\in\mathbf{R}^2, \|\theta\| = 1\Big\} \leqq$$

$$\leqq \sup\Big\{-(\varepsilon/\sigma^2)\varepsilon+\log\int\exp\big((\varepsilon/\sigma^2)\langle\theta, Y_1(t)\rangle\big)\,dP\colon \theta\in\mathbf{R}^2, \|\theta\| = 1\Big\}$$

for all $\varepsilon>0$.

Let n be chosen large enough such that we have $\varepsilon_n<\sigma^2/4$. Then

$$\int\exp\big((\varepsilon_n/\sigma^2)\langle\theta, Y_1(t)\rangle\big)\,dP =$$

$$= 1+(\varepsilon_n^2/2\sigma^4)\operatorname{Var}\langle\theta, Y_1(t)\rangle+\sum_{\nu=3}^{\infty}\frac{1}{\nu!}(\varepsilon_n/\sigma^2)^\nu\int\big(\langle\theta, Y_1(t)\rangle\big)^\nu\,dP\leqq$$

$$\leqq 1+\varepsilon_n^2/2\sigma^2+\sum_{\nu=3}^{\infty}\frac{1}{\nu!}(2\varepsilon_n/\sigma^2)^\nu \leqq 1+\varepsilon_n^2/2\sigma^2+8\varepsilon_n^3/\sigma^6, \quad \text{if} \quad \theta\in\mathbf{R}^2, \|\theta\| = 1.$$

Since

$$i_t(\varepsilon_n)\leqq-\varepsilon_n^2/\sigma^2+\log(1+\varepsilon_n^2/2\sigma^2+8\varepsilon_n^3/\sigma^6)\leqq-\varepsilon_n^2/2\sigma^2+8\varepsilon_n^3/\sigma^6,$$

we have

$$\varlimsup_{n\to\infty} i_t(\varepsilon_n)/\varepsilon_n^2 \leqq -1/2\sigma^2.$$

Combining this result with the first inequality we get the desired expansion.

Note added in proof. Theorem 2 can also be derived from [10] Theorem 2 by taking the set functions $f_{t,\theta}(x)=\theta_1\cos(tx)+\theta_2\sin(tx)$, $t\in S$, $\theta=(\theta_1, \theta_2)\in\mathbf{R}^2$, $\|\theta\|=1$, $x\in\mathbf{R}$.

References

[1] R. R. BAHADUR, On the asymptotic efficiency of tests and estimates, *Sankhyā*, **22** (1960), 229—252.

[2] S. CSÖRGŐ, Limit behaviour of the empirical characteristic function, *Ann. Probability*, 9 (1981), 130—144.

[3] S. CSÖRGŐ, Multivariate empirical characteristic functions, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **55** (1981), 203—229.

[4] S. CSÖRGŐ and V. TOTIK, On how long interval is the empirical characteristic function uniformly consistent?, *Acta Sci. Math.*, **45** (1983), 141—149.

[5] A. FEUERVERGER and R. A. MUREIKA, The empirical characteristic function and its applications, *Ann. Statist.*, **5** (1977), 88—97.

[6] P. GROENEBOOM, Large deviations and asymptotic efficiences, Mathematical Centre Tracts 118, Mathematical Centre, Amsterdam, 1980.

[7] H.-D. KELLER, Einige Untersuchungen zur empirischen charakteristischen Funktion und deren Anwendungen, Dissertation, Dortmund, 1979.

[8] M. B. MARCUS, Weak convergence of the empirical characteristic function, *Ann. Probability*, **9** (1981), 194—201.

[9] J. S. RAO, Bahadur efficiencies of some tests for uniformity on the circle, *Ann. Math. Statist.*, **43** (1972), 468—479.

[10] J. SETHURAMAN, On the probability of large deviations of families of sample means, *Ann. Math. Statist.*, **35** (1964), 1304—1316.

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